Spectro-Hierarchical Homogenization scheme for elasto-dynamic problems in periodic Cauchy materials

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Abstract

A spectro-hierarchical algorithm is proposed to determine an approximate solution to the elastodynamic problem for periodic heterogeneous materials. Such a homogenization scheme is derived by employing tools from perturbation theory in finite dimension used in the context of the celebrated Theorem of Nekhoroshev. In particular, it is shown how a classical algorithm based on a suitable hierarchy of harmonics can be implemented for the problem at hand, leading to the explicit construction of functions approximating the solution of the original problem with an error that is superexponentially small in the cell dimension. According to this approach, the fully homogenised model turns out to be naturally related to the "integrable case" of perturbation theory. Furthermore, all the featured constants are estimated explicitly. More importantly, a fully detailed bound of the threshold for the cell dimension is presented to ensure the validity of the theory.

Keywords: Microstuctured heterogeneous materials, transient elasto-dynamics, perturbative methods, Nekhoroshev theory.

1 Introduction

Heterogeneous materials, such as composites or bio-inspired materials, are increasingly attracting the attention of researchers due to their many applications in different fields, ranging from engineering to robotics to aerospace. Such materials are interesting because their mechanical behaviors can be designed by properly changing the shape, location, density, and distribution of the discrete phases (e.g., particles, fibers, voids, inclusions) embedded into a continuous matrix. Within this framework, homogenization techniques are essential to determine the mechanical behavior of periodically arranged materials that would require huge computational resources [1]. In recent years, different multi-scale homogenization schemes have been proposed to determine the equivalent static and dynamic mechanical properties of periodic heterogeneous materials. Besides classical homogenization approaches [2], the mechanical properties of a continuum, equivalent to the periodic heterogeneous materials, can be obtained via (i) asymptotic schemes [3–15], (ii)

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variational-asymptotic schemes [16–22], and (iv) other identification techniques. These latter techniques comprise analytical [23–34] and computational [35–54] approaches. Such methods have also been extended to describe the macroscopic behavior of heterogeneous materials in multiphysics problems involving thermo-mechanical [55–60], thermo-magneto-electro-elastic [61–64], and electro-mechanical [65–70] coupling effects. The elastic dynamic behavior of periodic heterogeneous materials is governed by elastodynamic PDEs of motion whose exact analytical and numerical solution is very challenging to be obtained. Here, we provide a novel spectro-hierarchical homogeneization algorithm to determine an approximate solution to the PDEs governing the elastodynamic problem for periodic heterogeneous materials. Perturbative approaches in the PDEs field are nowadays quite common, and many different techniques have been developed through the years to tackle a vast class of problems. For instance, the search for quasi periodic solutions in PDEs has required a remarkable effort to "build a bridge" between the celebrated results in finite dimension, such as the collection of techniques nowadays known as KAM Theory (after Kolmogorov, Arnol'd and Moser, see e.g. [71] and references therein) and their "equivalent" in infinite dimension. For a comprehensive review of the results and tools developed in the context of this "big leap" we refer, for instance, to [72] and [73] and references therein. Another example of this process is represented by the diagrammatic techniques described e.g. in [74, Chap. 8]. These works, originally set in finite dimension, have been successfully employed in many relevant infinite dimensional models, see e.g. [75] or [76, Chap. 12].

In a typical perturbative setting, the present work relies on the possibility to decompose an original initial value problem (IVP) into an infinite hierarchy of IVPs which have the advantage to be explicitly resolvable. Typically, this is achieved by exploiting the smallness of some characteristic parameter of the problem, with the aim to obtain the convergence of such a hierarchy to an "object" which could interpreted as the continuation of the solution in the unperturbed case. The earliest results in finite dimension have shown that, at least in general, this is far from being an easy task. It is sufficient to mention, for this purpose, the challenging problem represented by the well known *small divisors*. This phenomenon is not limited to the effects of resonances but it involves also other "artificial" terms, introduced, for instance, by the Cauchy bounds in the real-analytic context. All the above mentioned works rely on some key argument to face this difficulty: for instance, the Nash-Moser approach uses the speed of convergence of a quadratic method, whilst the diagrammatic techniques are able to track down some key compensations (cancellations) between those terms.

The works by Nekhoroshev [77], [78], another pillar of the stability theory in finite dimension (which has been extended to some cases in infinite dimension as well, see e.g. [79]), has suggested, however, that even in cases in which an obstruction to the convergence of the perturbative series exists, these could still be used to deduce remarkable information on the system stability. This is the case, for instance, when one looks for stability of a quite general class of nearly-integrable Hamiltonians in a whole open set of the phase space. In fact, with the use of a particularly clever geometric argument, Nekhoroshev has shown that (non-convergent) perturbative arguments can prove solutions to be stable over exponentially long times: in Littlewood's very own words: *"[...] while not eternity, this is a considerable slice of it"*.

Our aim is to show that, exactly in the same spirit as (the analytic part of) Nekhoroshev's work, a perturbative approach can construct solutions that, despite not exact, are "remarkably" close to be so. More precisely, it will be shown that the obtained functions are "superexponentially" close to a solution, in a sense that will be made precise in the main statement. The advantage lies in the full constructivity of the procedure, which may serve in explicit computations for models arising from applications.

The (classical) scheme employed in this work relies essentially on the well known decay of the harmonics of a periodic function, and on a suitable organisation of them in hierarchies, according to their size (from which the adjective spectro-hierarhical). This approach looks different in nature, however, from algorithms classically employed in asymptotic and variational-asymptotic homogenization theory, such as the one presented in e.g. [3] and [16]. It has to be stressed that the key ingredient of the proof and the very reason behind such an extremely encouraging error bound is the separation between the harmonics of a rapidly oscillating function and the exploitation of this property.

The work borrows ideas either from the "finite dimension world" or from the infinite dimension one. Mainly, the perturbative treatment of real-analytic functions and some key bounds (including the choice of the normalisation order leading to the superexponential estimate) are carried out along the lines of the masterful presentation by Giorgilli [80, Chap. 5]. We mention that a similar approach has led to another successful application of a similar finite-dimensional perturbative setting to PDEs, as in [81].

The real-analytic scenario is clearly a "paradigmatic" choice to create a more efficient comparison with the mentioned tools from Perturbation Theory but it still looks a viable option to treat, by employing suitable approximations (for instance by using truncated Fourier expansions), more general functions. On the other hand, by relying on the decay of the Fourier coefficients only, the spectro-hyerarchical approach is not limited, at least in principle, to the real-analytic case and extensions to more general function spaces may be possible, despite with an expected worsening of the corresponding error bound. The effectiveness of the spectro-hierarchical homogenization scheme here proposed, will be presented in the forthcoming paper [82], where several physical examples are analyzed.

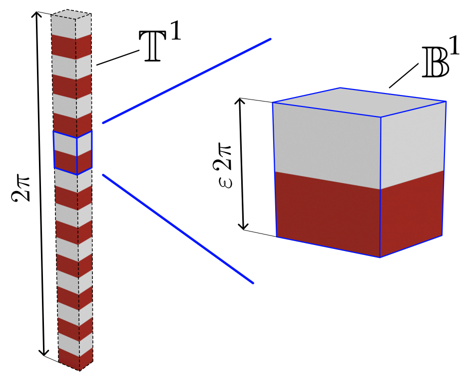

Figure 1: Heterogeneous materials with periodically distributes inclusions.

2 Model and main result

Let us consider the following elastodynamic Initial Value Problem (IVP)

$$\begin{cases} \mathcal{R}(\varepsilon^{-1}\boldsymbol{x})\partial_{tt}v_i(\boldsymbol{x},t) - \sum_{j=1}^{N}\partial_{x_j}\left[\sum_{h,k=1}^{N}\mathcal{C}_{ijhk}(\varepsilon^{-1}\boldsymbol{x})\partial_{x_k}v_h(\boldsymbol{x},t)\right] = \mathcal{G}_i(\boldsymbol{x},t) \\ v_i(\boldsymbol{x},0) = w_i(\boldsymbol{x}) \\ \partial_t v_i(\boldsymbol{x},0) = \tilde{w}_i(\boldsymbol{x}) \end{cases}, \quad i = 1,\dots,N, \quad (1)$$

where, $N \in \mathbb{N}$ is the problem dimension, v the displacement field, $\mathcal{R}$ the inertia term, $\mathcal{C}_{ijhk}$ the linear elastic constitutive tensor, $\mathcal{G}_i$ the body force term, $\boldsymbol{x} \in \mathbb{T}^N$, and $\varepsilon \in \mathbb{R}$ is a parameter

which is supposed to be "small" in a sense that will be specified later..
Let us stress that the assumption $\boldsymbol{x} \in \mathbb{T}^N$ does not imply any loss of generality as any other "characteristic length" different from 2π can be reduced to this case via a trivial rescaling and a suitable redefinition of the objects appearing in Eqn. (1).
The framework we are going to consider is a (standard) class of real-analytic functions defined on $\mathbb{T}^N \times [0,T]$, for some fixed $T>0$. For this purpose, given a parameter $\rho \in (0,1)$, we consider the complexified domain $\mathcal{D}_\rho := \mathbb{T}^N_\rho \times \mathcal{S}_\rho$, where

$$\mathbb{T}^N_\rho := \{\boldsymbol{x}' \in \mathbb{T}^N \, : \, \max_{j=1,\ldots,N} |\Im x'_j| \leq \rho\}, \qquad \mathcal{S}_\rho := \{t' \in \mathbb{C} \, : \, |t'-t| \leq \rho, \quad \forall t \in [0,T]\},$$

then we define as $\mathscr{H}_\rho$, the space of functions that are continuous on $\mathcal{D}_\rho$, holomorphic on its interior and real on $\mathcal{D}_0$. Specifically, the parameter ρ is commonly referred to as *analyticity radius of a function belonging to* $\mathscr{H}_\rho$. Similarly, we define $\mathscr{H}^r_\rho$ simply by replacing $\mathcal{D}_\rho$ with $\mathbb{T}^N_\rho$ and all the definitions given below are straightforwardly adopted to functions depending upon $\boldsymbol{x}$ only.
Given a function $F \in \mathscr{H}_\rho$, this can be expanded as $F(\boldsymbol{x},t) =: \sum_{\boldsymbol{\nu}\in\mathbb{Z}^N} f_{\boldsymbol{\nu}}(t) \exp(i\boldsymbol{\nu}\cdot\boldsymbol{x})$. Within this framework, it is customary to define (see e.g. [80]), the *Fourier norm* as

$$\|F(\boldsymbol{x},t)\|_\rho := \sum_{\boldsymbol{\nu}\in\mathbb{Z}^N} |f_{\boldsymbol{\nu}}(t)|_\rho \exp(|\boldsymbol{\nu}|\rho),$$

where $|f_{\boldsymbol{\nu}}(t)|_\rho := \sup_{t\in\mathcal{S}_\rho} |f_{\boldsymbol{\nu}}(t)|$ and $|\boldsymbol{\nu}| := |\nu_1| + \ldots + |\nu_N|$. Similarly, we shall set $|F(\boldsymbol{x},t)|_\rho := \sup_{(\boldsymbol{x},t)\in\mathcal{D}_\rho} |F(\boldsymbol{x},t)|$. As it is easy to check, the above defined $|F|_\rho \leq \|F\|_\rho$, see [80].
For any tensor-valued function $\mathcal{T}_{\sigma_1\sigma_2\ldots\sigma_n} \in \mathscr{H}_\rho$, the notation $\|\mathcal{T}_{\sigma_1\sigma_2\ldots\sigma_n}\|_\rho := \sum_{\sigma_i=1,\ldots,n} \|\mathcal{T}_{\sigma_1\sigma_2\ldots\sigma_n}\|_\rho$ will be used. In particular, in the presence of a function $\boldsymbol{F} := \{\mathcal{T}_\sigma\}_{\sigma=1,\ldots,N}$, one has $\|\boldsymbol{F}\|_\rho := \sum_{i=1,\ldots,n} \|F_i\|_\rho$. The same notation applies to the norm $|\cdot|_\rho$, i.e., $|\boldsymbol{F}|_\rho := \sum_{i=1,\ldots,n} |F_i|_\rho$.
It will be said that $F \in \mathscr{H}_\rho$ belongs to the class $\mathscr{F}_{a,b}$, for any $0 \leq a < b < +\infty$, if all the Fourier coefficients such that $|\boldsymbol{\nu}| \in [0,a) \cup (b,+\infty)$, satisfy $f_{\boldsymbol{\nu}}(t) \equiv 0$.

Hypothesis 2.1. *Let us suppose that:*

1. $\mathcal{R}(\boldsymbol{x}), \mathcal{C}_{ijhk}(\boldsymbol{x}), \mathcal{G}_i(\boldsymbol{x}), w_i(\boldsymbol{x}), \tilde{w}_i(\boldsymbol{x}) \in \mathscr{H}^r_{2\rho}$. *Furthermore,* $\mathcal{G}_i(\boldsymbol{x})$ *have null average for all* i.
2. *There exists* $r^- > 0$ *such that, for all* $\boldsymbol{x} \in \mathbb{T}$,
$$\mathcal{R}(\boldsymbol{x}) > r^-. \tag{2}$$
3. *The matrix obtained from* $\mathcal{C}_{ijhk}$ *for any given values of* i,h, *denoted below with* $\boldsymbol{\mathcal{C}}_{i\cdot h\cdot}$, *is positive definite. In particular, there exists a constant* c_λ *such that*
$$\boldsymbol{y}\cdot\boldsymbol{\mathcal{C}}_{i\cdot h\cdot}\boldsymbol{y}^\top \geq c_\lambda |\boldsymbol{y}|^2, \tag{3}$$
for all $\boldsymbol{y} \in \mathbb{R}^N$ *and all* $i,j = 1,\ldots,N$.
4. *The following bounds hold*
$$\|w_i(\boldsymbol{x})\|_{2\rho}, \|\tilde{w}_i(\boldsymbol{x})\|_{2\rho} \leq C^*. \tag{4}$$

Then we are able to prove the following:

Theorem 2.1. *Assume Hyp. 2.1. Let us choose* Γ *as*

$$\Gamma \geq \Gamma^* := \lceil 2/\rho \rceil, \tag{5}$$

and suppose that

$$w_i(\boldsymbol{x}), \tilde{w}_i(\boldsymbol{x}) \in \mathscr{F}_{0,\Gamma}. \tag{6}$$

Then there exists $\varepsilon_0 > 0$ *such that, for all* $\varepsilon \in (0, \varepsilon_0]$ *and satisfying the following condition*

$$\sigma := (\varepsilon\Gamma)^{-1} \in \mathbb{N}, \tag{7}$$

it is possible to construct a unique $O(1)$ *function* $v_i^{[<p]}(\boldsymbol{x}, t) \in \mathscr{H}_{\rho/2}$ *satisfying the initial conditions in (1) and such that the "error"* $v_i^{[p]} := v_i - v_i^{[<p]}$*, with* v_i *satisfying (1), is bounded as follows*

$$\max_{t\in[0,T]} \int_{\mathbb{T}^N} \left| \boldsymbol{v}^{[p]}(\boldsymbol{x}, t) \right|^2_{\varepsilon\rho/2} d\boldsymbol{x} \leq \tilde{\mathscr{S}} \exp\left[-\left(\tilde{\mathscr{E}} e^{\frac{\rho}{4\varepsilon}} \right)^{\frac{1}{2+N}} \right], \tag{8}$$

where $\tilde{\mathscr{S}}, \tilde{\mathscr{E}}$ *are* $O(1)$ *constants.*

Remark 2.1. *The meaning of the "superexponentially small remainder" is made precise by expression (8). Although* $v_i^{[<p]}(\boldsymbol{x}, t)$ *is "just" an approximate solution, it is clearly how dramatically the precision of this approximation increases as the size of* ε *is reduced.*

The rest of the paper is devoted to the proof of Theorem 2.1. This will be achieved by transforming Eqn. (1) into a hierarchy of explicit PDEs which can be straightforwardly resolved in a suitable Fourier space. As it is common in this kind of arguments, the proof is organised in two steps: a "formal part", in which a suitable algorithm apt to derive the desired solution is set up, and a consequent "quantitative part", in which the (non-trivial) problem of bounding the constructed solutions is addressed.
The mentioned algorithm is outlined in the flow chart depicted in Fig. 2 together with the essential references to quantitative aspects of it.

3 Formal scheme

Let us expand the functions appearing in Eqn. (1) as follows

$$\mathcal{G}_i = \sum_{s=1}^{p} \lambda^s G_i^{[s]} \tag{9}$$

$$\mathcal{R} = \sum_{s=0}^{p} \lambda^s R^{[s]} \tag{10}$$

$$\mathcal{C}_{ijhk} = \sum_{s=0}^{p} \lambda^s C_{ijhk}^{[s]} \tag{11}$$

where $\lambda \in \mathbb{R}$ is an "ordering parameter" and it will be thought as "one" throughout the whole process, as it is customary in Perturbation Theory. More precisely, we shall set, for all $s = 1, \ldots, p-1$

$$G_i^{[s]} = \sum_{(s-1)\Gamma < |\nu| \leq s\Gamma} g_\nu^{(i)} e^{i\nu\cdot x} \tag{12}$$

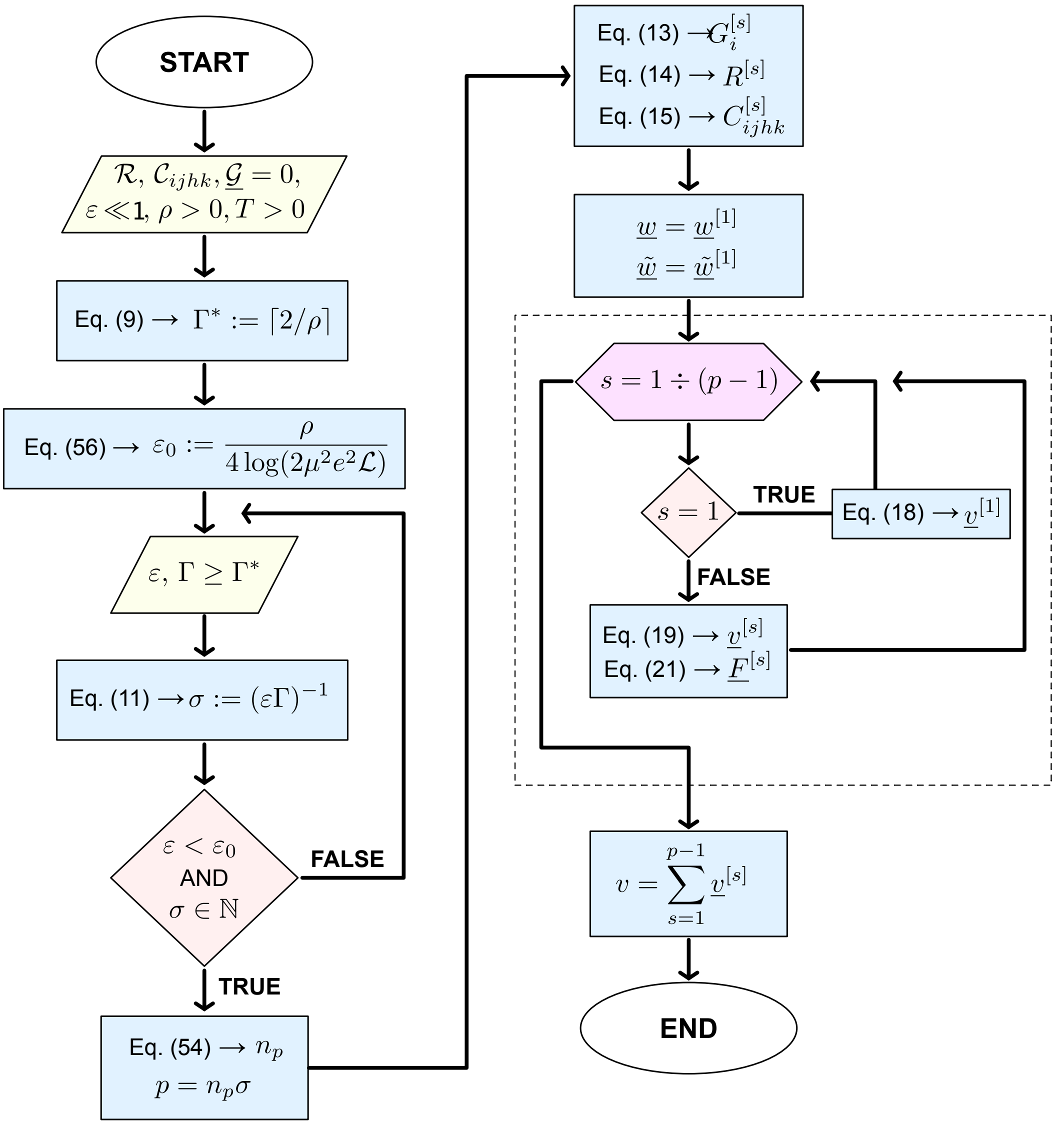

Figure 2: Flow chart for Spectro-Hierarchical Homogenization scheme.

whilst the p−th element of the sequence has the meaning of a "remainder"

$$G_i^{[p]} := \mathcal{G}_i - \sum_{s=1}^{p-1} \lambda^s G_i^{[s]}.$$

The same notation applies to $\mathcal{R}$, $\mathcal{C}_{ijhk}$ and v_i.
As a consequence of Eqn. (12), every term "labelled" with λ^s, for all $s = 1, \ldots, p-1$, belongs to the class $\mathscr{F}_{(s-1)\Gamma, s\Gamma}$. Expansions of the form reported in Eqn. (12) are typical in Celestial Mechanics, and such a technique goes even back to Poincaré, see [80], [83]. In general, the aim is to exploit a key property of the real-analytic functions class: the exponential decay of (a suitable norm of) the Fourier coefficients (this well known feature will be recalled later on).
We stress that, unlike Eqns. (10) and (11), the expansion of Eqn. (9) starts from the first order in λ as we have supposed to have a zero-averaging source (body force) $\boldsymbol{\mathcal{G}}$.
The described expansions allow to transform Eqn. (1), into a hierarchy of *explicit* equations, as

stated in the next

Proposition 3.1. *Let us consider the following expansion*

$$v_i(x,t) = \sum_{s=1}^{p} \lambda^s v_i^{[s]}(x,t), \tag{13}$$

then Eqn. (1) is equivalent to the following set of IVPs for all $i = 1,\dots,N$,

$$\begin{cases} \mathcal{R}^{[0]}\partial_{tt}v_i^{[1]} - \displaystyle\sum_{j,h,k=1}^{N} \mathcal{C}_{ijhk}^{[1]}\partial_{x_j x_k} v_h^{[1]} = G_i^{[1]} \\ v_i^{[1]}(\boldsymbol{x},0) = w_i(x) \\ \partial_t v_i^{[1]}(\boldsymbol{x},0) = \tilde{w}_i(x) \end{cases} \tag{14}$$

and

$$\begin{cases} R^{[0]}\partial_{tt}v_i^{[s]} - \displaystyle\sum_{j,h,k=1}^{N} C_{ijhk}^{[0]}\partial_{x_j x_k} v_h^{[s]} = G_i^{[s]} + F_i^{[s]} \\ v_i^{[s]}(\boldsymbol{x},0) \equiv 0 \\ \partial_t v_i^{[s]}(\boldsymbol{x},0) \equiv 0 \end{cases} \tag{15}$$

for all $s = 2,\dots,p-1$, whilst the "remainder" is given by

$$\left(\sum_{s=1}^{p} \lambda^s R^{[s]}\right)\partial_{tt}v_i^{[p]} - \sum_{j}^{N}\partial_{x_j}\sum_{h,k=1}^{N}\left(\sum_{s=1}^{p}\lambda^s C_{ijhk}^{[s]}\right)\partial_{x_k}v_h^{[p]} = G_i^{[p]} + F_i^{[p]} \tag{16}$$

subject to $v_i^{[p]}(\boldsymbol{x},0) = \partial_t v_i^{[p]}(\boldsymbol{x},0) \equiv 0$, where we have set

$$F_i^{[s]} = \begin{cases} \displaystyle\sum_{r=1}^{s-1}\left[-R^{[r]}\partial_{tt}v_i^{[s-r]} + \sum_{j=1}^{N}\partial_{x_j}\left(\sum_{h,k=1}^{N} C_{ijhk}^{[r]}\partial_{x_k}v_h^{[s-r]}\right)\right] & \text{if} \quad s = 2,\dots,p-1 \\ \displaystyle\sum_{r=1}^{p-1}\left[-R^{[r]}\partial_{tt}v_i^{[p-r]} + \sum_{j=1}^{N}\partial_{x_j}\left(\sum_{h,k=1}^{N} C_{ijhk}^{[r]}\partial_{x_k}v_h^{[p-r]}\right)\right] + & \\ \displaystyle -\sum_{q=0}^{p-1}\lambda^q\sum_{r=0}^{p-q-1}\left[R^{[p-r]}\partial_{tt}v_i^{[r+q]} - \sum_{j=1}^{N}\partial_{x_j}\left(\sum_{h,k=1}^{N} C_{ijhk}^{[p-r]}\partial_{x_k}v_h^{[r+q]}\right)\right] & \text{if} \quad s = p \end{cases} \tag{17}$$

Remark 3.1. *Let us write down explicitly the first terms of the sequence $F_i^{[s]}$ (they will be useful later).*

$$F_i^{[2]} = -R^{[1]}\partial_{tt}v_i^{[1]} + \sum_{j=1}^{N}\partial_{x_j}\left(\sum_{h,k=1}^{N} C_{ijhk}^{[1]}\partial_{x_k}v_h^{[1]}\right)$$

$$F_i^{[3]} = -R^{[2]}\partial_{tt}v_i^{[1]} - R^{[1]}\partial_{tt}v_i^{[2]} + \sum_{j=1}^{N}\partial_{x_j}\left[\sum_{h,k=1}^{N}\left(C_{ijhk}^{[2]}\partial_{x_k}v_h^{[1]} + C_{ijhk}^{[1]}\partial_{x_k}v_h^{[2]}\right)\right]$$

It is immediate to realise that any $F_i^{[s]}$ depends on $v_i^{[0]}, v_i^{[1]},\dots,v_i^{[s-1]}$ (other than the known $R^{[s]}, C_{ijhk}^{[s]}$), i.e. every equation possesses an explicit structure, as anticipated. This clarifies the claimed constructive feature of the algorithm.

Proof. (Sketch). Via a straightforward substitution of the expansions in Eqns. (9), (10), (11) and (13) into Eqn. (1) then equating the coefficients of λ^s, for all $s = 1, 2, \dots, p-1$. These coefficients are found via standard *à la Cauchy* product formulae. For instance, as for the first term of Eqn. (1), one has

$$\left(\sum_{s=0}^{p} \lambda^s R^{[s]}\right)\left(\sum_{s=1}^{p} \lambda^s \partial_{tt} v_i^{[s]}\right) = \sum_{s=1}^{p} \lambda^s \left(\sum_{r=0}^{s-1} R^{[r]} \partial_{tt} v_i^{[s-r]}\right) + \lambda^p \sum_{s=1}^{p} \lambda^s \left(\sum_{r=0}^{p-s} R^{[p-r]} \partial_{tt} v_i^{[r+s]}\right).$$

□

4 Preliminary tools and results for the proof

The following proposition contains a key result for our purposes. As anticipated, the exponentially decaying behaviour of the Fourier coefficients of a real-analytic function is a well known fact. However, such a bound turns out to be far too pessimistic when dealing with functions, like e.g. $\mathcal{R}$, which exhibit a "ε^{-1}-fast" dependence upon their argument. More precisely, such a decay will be "ε-slow" and the argument shown in Sec. 5 would fail by a *tout court* employment of such an exponential decay, unless ε is not small. The key ingredient consists in taking into account that the non-zero Fourier coefficients of such a functions are more "separated" on $\mathbb{Z}^N$ the more ε gets smaller. For instance, the non-zero harmonics of the function $f(z, \varepsilon)$ in the one-dimensional case described in Fig. 3, which occur for even values of n if $\varepsilon = 1$ (panels (A,C)), are "shifted" to $n = 0, 20, 40, 60, \dots$ if $\varepsilon = 1/10$ (panels (B,D)).

Proposition 4.1. *Suppose that* $\varepsilon\Gamma \leq 1$, *define*

$$\mathfrak{I} := \{\lceil 1/(\varepsilon\Gamma)\rceil, \lceil 2/(\varepsilon\Gamma)\rceil, \lceil 3/(\varepsilon\Gamma)\rceil, \dots\} \subset \mathbb{N}$$

and recall Hypothesis 2.1, (1). Then, the following bounds hold

$$\left\|R^{[s]}(\varepsilon^{-1}\boldsymbol{x})\right\|_{\varepsilon\rho}, \left\|C^{[s]}_{ijhk}(\varepsilon^{-1}\boldsymbol{x})\right\|_{\varepsilon\rho} \leq \mathcal{A}\tilde{\alpha}_s \qquad ; \qquad \tilde{\alpha}_s = \begin{cases} \alpha^s & s \in \mathfrak{I} \\ 0 & \textit{otherwise} \end{cases} \tag{18}$$

where

$$\mathcal{A} = \mu \left(\frac{e^{\Gamma\rho} - 1}{e^{\rho} - 1}\right)\left(\frac{2N}{\rho}\right)^N e^{-(N-\rho/2)}, \qquad \alpha := e^{-\Gamma\rho/2}, \tag{19}$$

having set

$$\mu := \max\left\{|\mathcal{R}(\boldsymbol{x})|_{\rho}, |\mathcal{C}_{ijhk}(\boldsymbol{x})|_{\rho}\right\}. \tag{20}$$

Proof. We will carry out the proof for $\mathcal{R}$, the one for $\mathcal{C}_{ijhk}$ being analogous.
By denoting with $r_{\boldsymbol{\nu}}$ the coefficients of the Fourier expansion of $\mathcal{R}$, these satisfy the well known bound (see e.g. [80] for a proof)

$$|r_{\boldsymbol{\nu}}| \leq \left|\mathcal{R}(\varepsilon^{-1}\boldsymbol{x})\right|_{2\varepsilon\rho} e^{-2\varepsilon|\boldsymbol{\nu}|\rho}.$$

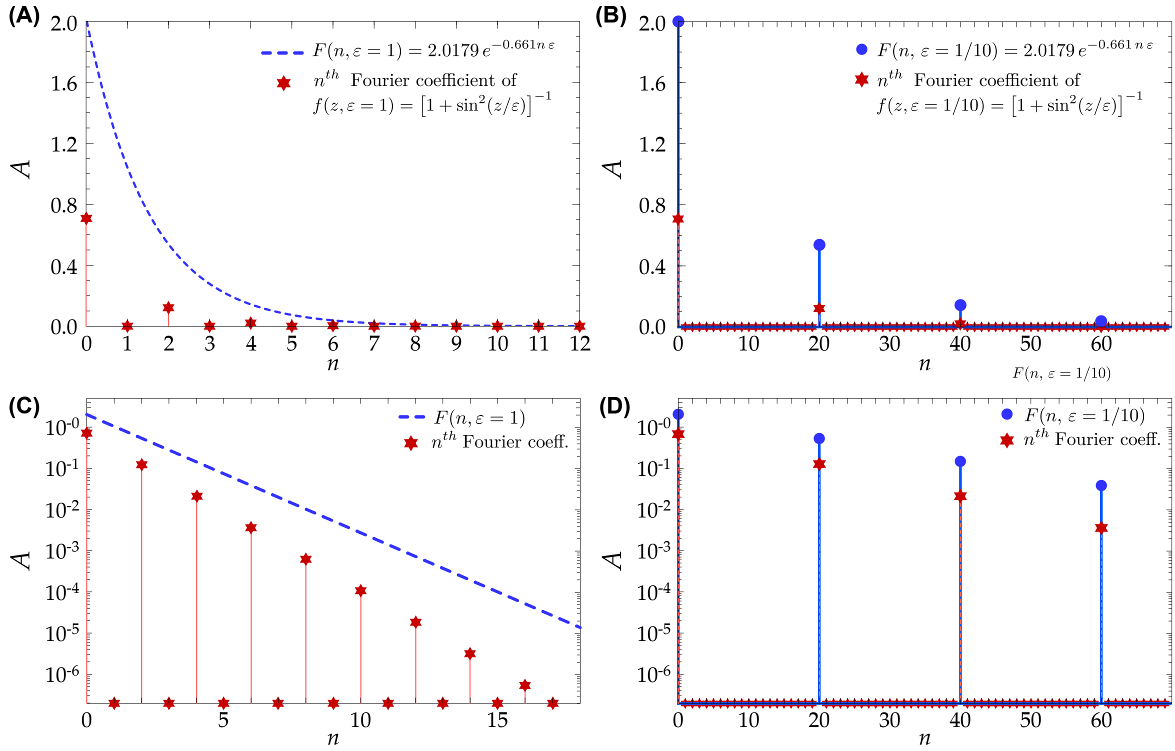

Figure 3: Decay of the Fourier harmonics either in the linear (A,B) or in the logarithmic (C,D) scale for the paradigmatic real-analytic function $f(z, \varepsilon) = [1 + \sin(z/\varepsilon)^2]^{-1}$ with respect to the bound computed via Prop. 4.1 for two different values of ε. The right panel (B) shows the separation between the non-vanishing harmonics of f in case of "large" ε, which plays a key role in the proof of the main result.

Hence, this leads to

$$\begin{aligned}
\left\| R^{[s]}(\varepsilon^{-1}\boldsymbol{x}) \right\|_{\varepsilon\rho} &\leq \left|\mathcal{R}(\varepsilon^{-1}\boldsymbol{x})\right|_{2\varepsilon\rho} \sum_{\substack{\boldsymbol{m}\in\mathbb{Z}^N \\ \varepsilon\boldsymbol{m}\in\mathbb{Z}^N \\ \varepsilon(s-1)\Gamma<\varepsilon|\boldsymbol{m}|\leq\varepsilon s\Gamma}} e^{-\varepsilon|\boldsymbol{m}|\rho} \\
&\leq \left|\mathcal{R}(\varepsilon^{-1}\boldsymbol{x})\right|_{2\varepsilon\rho} [\mathrm{Card}\,(\mathfrak{A}_l)] \left(\sum_{l=(s-1)\Gamma+1}^{s\Gamma} e^{-l\rho} \right) \\
&\leq \left|\mathcal{R}(\varepsilon^{-1}\boldsymbol{x})\right|_{\rho} [\mathrm{Card}\,(\mathfrak{A}_{s\Gamma})] e^{-\Gamma s\rho} (e^{\rho} - 1)^{-1} (e^{\Gamma\rho} - 1)
\end{aligned}$$

where $\mathfrak{A}_l := \{\boldsymbol{\nu} \in \mathbb{Z}^N : |\boldsymbol{\nu}| = l\}$ and we have used that $\varepsilon < 1/2$, see, e.g. (51). The sum appearing in the last inequality can be bounded by $(s\Gamma + 1)^N$, hence the proof is complete by using the bound

$$(x + 1)^N e^{-\rho x} \leq (2\rho^{-1}N)^N e^{-(N-\rho/2)} e^{-\rho x/2}, \tag{21}$$

which holds, in particular, for all $x \geq 0$, $N \geq 1$ and $\rho \in (0, 1]$, where one can set $x \leftarrow s\Gamma$. □

Proposition 4.2. *Let us consider the following Cauchy problem*

$$\begin{cases} R^{[0]}\partial_{tt}w_i - \sum_{j,h,k=1}^{N} C^{[0]}_{ijhk}\partial_{x_jx_k}w_h = \mathcal{N}_i \\ w_i(\boldsymbol{x},0) = \mathcal{F}^{(1)}_i(\boldsymbol{x}) \\ \partial_t w_i(\boldsymbol{x},0) = \mathcal{F}^{(2)}_i(\boldsymbol{x}) \end{cases} \qquad i = 1,\dots,N, \tag{22}$$

where, given $\mathcal{I},\mathcal{H} \subseteq \mathbb{Z}^N$*,*

$$\mathcal{N}_i =: \sum_{\boldsymbol{\nu}\in\mathcal{I}} g_{\boldsymbol{\nu},i}(t)e^{i\boldsymbol{\nu}\cdot\boldsymbol{x}}, \qquad \mathcal{F}^{(j)}_i =: \sum_{\boldsymbol{\nu}\in\mathcal{H}} f^j_{\boldsymbol{\nu},i}e^{i\boldsymbol{\nu}\cdot\boldsymbol{x}},$$

i.e. the Fourier coefficients of $\mathcal{N}_i$ *and of* $\mathcal{F}^{(j)}_i$ *do not vanish (identically) on* $\mathcal{I}$ *and* $\mathcal{H}$*, respectively. Suppose that, for some* $\tilde{\rho} \le \rho$ *and* $\tilde{\beta},\hat{\gamma} \in \mathbb{R}$*, the following bounds hold*

$$\|\boldsymbol{\mathcal{N}}\|_{\hat{\rho}} \le \hat{\beta}, \qquad 2\left\|\boldsymbol{F}^{(j)}\right\|_{\hat{\rho}} \le \hat{\gamma}. \tag{23}$$

Then, for all $T \in (0,+\infty)$*, there exists a unique solution of the Cauchy problem (22) whose Fourier coefficients satisfy, for all* $i = 1,\dots,N$*,*

$$\max_{t\in[0,T]} |c_{\boldsymbol{\nu},i}(t)| \le N\left\{\hat{\gamma}/r^- + \mu T\hat{\beta}/c_\lambda\right\}e^{-|\boldsymbol{\nu}|\hat{\rho}}, \qquad \forall\boldsymbol{\nu} \in (\mathcal{I}\cup\mathcal{H}), \tag{24}$$

and $c_{\boldsymbol{\nu},i}(t) \equiv 0$ *for all* $\boldsymbol{\nu} \in \mathbb{Z}^N \setminus (\mathcal{I}\cup\mathcal{H})$*. Recall that* c_λ *has been defined in Eqn. (3).*

Remark 4.1. *The statements holds also in the case in which either* $\mathcal{I}$ *or* $\mathcal{H}$ *are empty.*

Proof. Let us expand $w_i(\boldsymbol{x},t)$ in Fourier series with coefficients $c_{\boldsymbol{\nu},i}(t)$ and substitute it into problem (22), this yields

$$-\sum_{j,h,k=1}^{N} C^{[0]}_{ijhk}\partial_{x_jx_k}w_h = \sum_{\boldsymbol{\nu}\in\mathcal{I}}\left[\left(\sum_{j,k=1}^{N}\nu_j\nu_k C^{[0]}_{ij1k}\right)c_{\boldsymbol{\nu},1} + \dots + \left(\sum_{j,k=1}^{N}\nu_j\nu_k C^{[0]}_{ijNk}\right)c_{\boldsymbol{\nu},N}\right]e^{i\boldsymbol{\nu}\cdot\boldsymbol{x}}$$

Hence, by defining the matrix whose rows are indexed by i and columns by h,

$$\Delta_{\boldsymbol{\nu}} := \left\{\sum_{j,k=1}^{N}\nu_j\nu_k C^{[0]}_{ijhk}\right\}_{ih}$$

and the (column) vector $\boldsymbol{c}_{\boldsymbol{\nu}} := (c_{\boldsymbol{\nu},1}, c_{\boldsymbol{\nu},2},\dots,c_{\boldsymbol{\nu},N})^\top$ (the same notation will be used to define $\boldsymbol{g}_{\boldsymbol{\nu}}$ and $\boldsymbol{f}^{1,2}_{\boldsymbol{\nu}}$), problem (22) reads as

$$\begin{cases} R^{[0]}\ddot{\boldsymbol{c}}_{\boldsymbol{\nu}} + \Delta_{\boldsymbol{\nu}}\boldsymbol{c}_{\boldsymbol{\nu}} = \boldsymbol{g}_{\boldsymbol{\nu}} \\ \boldsymbol{c}_{\boldsymbol{\nu}}(0) = \boldsymbol{f}^1_{\boldsymbol{\nu}} \\ \dot{\boldsymbol{c}}_{\boldsymbol{\nu}}(0) = \boldsymbol{f}^2_{\boldsymbol{\nu}} \end{cases} \tag{25}$$

Due to the assumptions on $\mathcal{C}_{ijhk}$, the matrix $\Delta_{\boldsymbol{\nu}}$ is symmetric and positive-definite. Hence, there exists an orthogonal matrix P such that

$$P^T\Delta_{\boldsymbol{\nu}}P = \Lambda \qquad ; \qquad \Lambda = \mathrm{diag}(\lambda_1,\dots,\lambda_N)$$

with $\lambda_j \in \mathbb{R}$ being the (real and strictly positive) eigenvalues of $\Delta_{\boldsymbol{\nu}}$. Hence, by defining (implicitly) the vector $\boldsymbol{z}_{\boldsymbol{\nu}}$ as

$$\boldsymbol{c}_{\boldsymbol{\nu}} =: P\boldsymbol{z}_{\boldsymbol{\nu}}, \tag{26}$$

and left-multiplying both sides of first equation of problem (25) by $P^\top$ we get

$$\ddot{\boldsymbol{z}}_{\boldsymbol{\nu}} + \tilde{\Lambda}\boldsymbol{z}_{\boldsymbol{\nu}} = \tilde{P}^\top \boldsymbol{g}_{\boldsymbol{\nu}},$$

where $\tilde{\Lambda} := (R^{[0]})^{-1}\Lambda$ and $\tilde{P} := (R^{[0]})^{-1}P$. In this way, the task of finding a solution of problem (25), is reduced to the resolvability of the N uncoupled Cauchy problems

$$\begin{cases} \ddot{z}_{\boldsymbol{\nu},i}(t) + \tilde{\lambda}_i z_{\boldsymbol{\nu},i}(t) = \tilde{\boldsymbol{x}}_i \cdot \boldsymbol{g}_{\boldsymbol{\nu}}(t) \\ z_{\boldsymbol{\nu},i}(0) = \tilde{\boldsymbol{x}}_i \cdot \boldsymbol{c}_{\boldsymbol{\nu}}(0) \\ \dot{z}_{\boldsymbol{\nu},i}(0) = \tilde{\boldsymbol{x}}_i \cdot \dot{\boldsymbol{c}}_{\boldsymbol{\nu}}(0) \end{cases}, \qquad i = 1, \ldots, N,$$

where $\tilde{\boldsymbol{x}}_i$ are the columns of $\tilde{P}$. Their solution is easily found as

$$z_{\boldsymbol{\nu},i}(t) = (\tilde{\boldsymbol{x}}_i \cdot \boldsymbol{c}_{\boldsymbol{\nu}}(0)) \cos\left(t\sqrt{\tilde{\lambda}_i}\right) + (\tilde{\boldsymbol{x}}_i \cdot \dot{\boldsymbol{c}}_{\boldsymbol{\nu}}(0)) \sin\left(t\sqrt{\tilde{\lambda}_i}\right) + \tilde{\lambda}_i^{-\frac{1}{2}} \int_0^t (\tilde{\boldsymbol{x}}_i \cdot \boldsymbol{g}_{\boldsymbol{\nu}}(s)) \sin\left[(t-s)\sqrt{\tilde{\lambda}_i}\right].$$

Let us now recall definition (20) and notice that taking the average of bound (2) one has, in particular, $R^{[0]} > r^-$. By using bounds (23), definition (26), and the fact that the vectors $\tilde{\boldsymbol{x}}_i$ are orthonormal, the bound (24) easily follows. □

Proposition 4.3. *Consider Eqn. (1) with $w_i(x) \equiv 0$ then suppose that*

$$\|\boldsymbol{\mathcal{G}}\|_{\tilde{\rho}} \leq \tilde{\beta}, \tag{27}$$

for some $\tilde{\rho} \leq \rho$ and $\tilde{\beta} \in \mathbb{R}$. Then we have

$$\max_{t\in[0,T]} \int_{\mathbb{T}^N} |\boldsymbol{v}(x,t)|_{\tilde{\rho}}^2 \, d\boldsymbol{x} \leq 4(2\pi)^N T^3 (r^-)^{-1} \exp(1/r^-)\tilde{\beta}^2.$$

Proof. The proof of this statement uses tools and arguments based on those described in [84, Chap. 10].
As it is easy to check, the homogeneous version of problem (1) possesses the following *energy* function

$$\mathcal{E} := \frac{1}{2}\int_{\mathbb{T}^N} \left(\mathcal{R}(\varepsilon^{-1}\boldsymbol{x})\partial_t \boldsymbol{v} \cdot \partial_t \boldsymbol{v} + \sum_{i,j,h,k=1}^N \mathcal{C}_{ijhk}(\varepsilon^{-1}\boldsymbol{x})\partial_{x_k} v_h \partial_{x_j} v_i \right) d\boldsymbol{x},$$

which gives

$$\dot{\mathcal{E}} := \sum_{i=1}^N \left[\int_{\mathbb{T}^N} \left(\mathcal{R}(\varepsilon^{-1}\boldsymbol{x})\partial_{tt} v_i - \sum_{j=1}^N \partial_{x_j} \sum_{h,k=1}^N \mathcal{C}_{ijhk}(\varepsilon^{-1}\boldsymbol{x})\partial_{x_k} v_h \right) \partial_t v_i d\boldsymbol{x} \right].$$

In particular, if $\boldsymbol{v}(x,t)$ satisfies the homogeneous problem (1) with $w_i(x) \equiv 0$ for all $i = 1, \ldots, N$, then $\mathcal{E}(t) = 0$ for all t. On the other hand, if the non-homogeneous problem is considered, one has

$$\mathcal{E}(t) = \int_0^t ds \int_{\mathbb{T}^N} \boldsymbol{\mathcal{G}}(\boldsymbol{x}, s) \cdot \partial_t \boldsymbol{v}(\boldsymbol{x}, s) d\boldsymbol{x}.$$

As a consequence, the following bound holds

$$|\mathcal{E}(t)| \leq \int_0^t ds \left(\int_{\mathbb{T}^N} |\boldsymbol{\mathcal{G}}|^2 d\boldsymbol{x}\right)^{\frac{1}{2}} \left(\int_{\mathbb{T}^N} |\partial_t \boldsymbol{v}|^2 d\boldsymbol{x}\right)^{\frac{1}{2}} \leq \frac{2}{\tilde{\mu}} \int_0^t ds \int_{\mathbb{T}^N} |\boldsymbol{\mathcal{G}}|^2 d\boldsymbol{x} + \frac{\tilde{\mu}}{2} \int_0^t ds \int_{\mathbb{T}^N} |\partial_t \boldsymbol{v}|^2 d\boldsymbol{x},$$

where the elementary inequality $2ab \leq (2/\tilde{\mu})a^2 + (\tilde{\mu}/2)b^2$ has been used, being $\tilde{\mu}$ a real, strictly positive, arbitrary parameter to be chosen later. On the other hand, by using the assumptions on $\mathcal{R}$ and the positive-definiteness of $\mathcal{C}_{ijhk}$ one can write

$$|\mathcal{E}(t)| \geq \frac{r^-}{2} \int_{\mathbb{T}^N} |\partial_t \boldsymbol{v}|^2 d\boldsymbol{x}.$$

Hence, by defining

$$\Phi(t) := \frac{1}{2} \int_{\mathbb{T}^N} |\partial_t \boldsymbol{v}|^2 d\boldsymbol{x}, \qquad \hat{U}(t) := \frac{2}{\tilde{\mu}} \int_0^t ds \int_{\mathbb{T}^N} |\boldsymbol{\mathcal{G}}|^2 d\boldsymbol{x},$$

the bounds above yield

$$\Phi(t) \leq U(t) + \lambda \int_0^t \Phi(s) ds, \tag{28}$$

where $U(t) := (r^-)^{-1}\hat{U}(t)$ and $\lambda := \tilde{\mu}/r^-$. Moreover, $U(t)$ can be bounded as follows

$$U(t) \leq \frac{2T}{\tilde{\mu} r^-} \int_{\mathbb{T}^N} \|\boldsymbol{\mathcal{G}}\|_{\tilde{\rho}}^2 \, d\boldsymbol{x} \leq \frac{2T}{\tilde{\mu} r^-} (2\pi)^N \tilde{\beta}^2,$$

the latter by using (27). Hence, the previous estimate and the Gronwall Lemma applied to (28), yield

$$\int_{\mathbb{T}^N} |\partial_t \boldsymbol{v}|^2 d\boldsymbol{x} \leq 4T^2 (r^-)^{-1} (2\pi)^N \exp(1/r^-) \tilde{\beta}^2,$$

in which the choice $\tilde{\mu} = 1/T$ has been made.
The proof is complete by using the elementary bound: $\max_{t\in[0,T]} |\boldsymbol{v}|^2 \leq T \max_{t\in[0,T]} |\partial_t \boldsymbol{v}|^2$. □

5 Quantitative estimates

Lemma 5.1. *Fix $p \in \mathbb{N}$ and consider the sequence $\{d_s\}$ defined by*

$$d_s := s/(2p), \tag{29}$$

and suppose that definition (7) holds.
Then the solution $v_i^{[s]}$ satisfies the following bound

$$\left\| v_i^{[s]} \right\|_{(1-d_s)\varepsilon\rho} \leq \gamma_s, \tag{30}$$

with

$$\gamma_{j\sigma+k} = \tilde{\mathcal{A}}_p \chi^k \left\{ \chi^{j\sigma} + \mathcal{B}_p \alpha^\sigma \sum_{l=1}^{j} \chi^{\sigma(j-l)} \left((1+\mathcal{B}_p)\alpha^\sigma\right)^{l-1} \right\}, \tag{31}$$

where $\tilde{\mathcal{A}}_p$ and $\mathcal{B}_p$ are positive constants (see definitions (38) and (40) for their explicit expression) and the index $s \in \mathbb{N}$ has been written as $s = j\sigma + k$ for all $j \geq 0$ and all $k = 1, \ldots, \sigma$.

Proof. By following the approach of the proof of Lemma 5.3. of [80, Chap. 5], let us firstly notice that, by Hyp. 2.1 (1) and (4), then using Prop. 4.2, one has

$$\left\|v_i^{[1]}\right\|_{(1-d_1)\varepsilon\rho} \leq \left\|v_i^{[1]}\right\|_{\varepsilon\rho} \leq NC^* \left\{\frac{1}{r^-}+\frac{\mu T}{c_\lambda}\right\} \sum_{\boldsymbol{\nu}\in\mathbb{Z}^N} e^{-(2-\varepsilon)\rho|\boldsymbol{\nu}|} \leq 2^{N+1} NC^* \left\{\frac{1}{r^-}+\frac{\mu T}{c_\lambda}\right\} =: \mathcal{K}_1, \tag{32}$$

where the inequality $(e^\rho - e^{-\rho})/(e^\rho - 1) \leq 2$ has been employed.
Hence, let us suppose that bound (30), in which we have set $s = r$, holds true for all $r < s$, then proceed by induction.
First of all, again by Hyp. 2.1, one has

$$\left\|G_i^{[s]}\right\| \leq \mathcal{T}\chi^s \tag{33}$$

where

$$\mathcal{T} := \left|\boldsymbol{G}^{[s]}(\boldsymbol{x},t)\right|_{2\rho} \left(\frac{e^{\Gamma\rho}-1}{e^\rho-1}\right)\left(\frac{2N}{\rho}\right)^N e^{-(N-\rho/2)}, \qquad \chi := e^{-\Gamma\rho/2}. \tag{34}$$

Let us now consider the expression of $F_i^{[s]}$ defined in (17) for $s < p$. As for the first term of the latter, let us notice that, after a Cauchy bound and (18), one has

$$\left\|\partial_{tt} v_i^{[s-r]}\right\|_{(1-d_{s-1/2})\varepsilon\rho} \leq \frac{2}{\varepsilon^2\rho^2(d_{s-1/2}-d_{s-r})^2}\gamma_{s-r}, \tag{35}$$

for all $i = 1, \ldots, N$. On the other hand, the second term is bounded similarly from (18), and more precisely

$$\begin{aligned}&\left\|\sum_{j,h,k=1}^N \left[\partial_{x_j}\mathcal{C}_{ijhk}^{[r]}\partial_{x_k}v_h^{[s-r]} + \mathcal{C}_{ijhk}^{[r]}\partial_{x_j x_k}v_h^{[s-r]}\right]\right\|_{(1-d_{s-1/2})\varepsilon\rho} \leq \\ &\leq \frac{\mathcal{A}N^3}{\varepsilon^2\rho^2}\left(\frac{1}{d_{s-1/2}(d_{s-1/2}-d_{s-r})} + \frac{1}{(d_{s-1/2}-d_{s-r})^2}\right)\tilde{\alpha}_s\gamma_{s-r}\end{aligned} \tag{36}$$

Bounds (33), (35) and (36) imply

$$\left\|G_i^{[s]} + F_i^{[s]}\right\|_{(1-d_{s-1/2})\varepsilon\rho} \leq \mathcal{T}\chi^s + \frac{4\mathcal{A}N^3}{\varepsilon^2\rho^2}\sum_{r=1}^{s-1}\frac{\tilde{\alpha}_s\gamma_{s-r}}{(d_{s-1/2}-d_{s-r})^2}. \tag{37}$$

From definition (29) we notice that $(d_{s-1/2} - d_{s-r})^{-2} \leq 16p^2$, as $r \geq 1$. Hence, by using the latter in bound (37) and using Prop. 4.2 with $\mathcal{N}_i := G_i^{[s]} + F_i^{[s]}$ and $\mathcal{F}_i^{1,2} \equiv 0$, we obtain

$$|c_{\boldsymbol{\nu},i}^{[s]}| \leq \mu NTc_\lambda^{-1}\left(\mathcal{T}\chi^s + 64\mathcal{A}N^3p^2(\varepsilon\rho)^{-2}\sum_{r=1}^s\tilde{\alpha}_s\gamma_{s-r}\right)e^{-|\boldsymbol{\nu}|(1-d_{s-1/2})\varepsilon\rho}.$$

Hence, by using the inequality

$$\sum_{(s-1)\Gamma\leq|\boldsymbol{\nu}|\leq s\Gamma} e^{-|\boldsymbol{\nu}|(d_s-d_{s-1/2})\varepsilon\rho} \leq \left(\frac{8N(p-1)}{\varepsilon\rho}\right)^N e^{1-N}$$

where we have used $(\varepsilon\rho \le 4p)$, and defining

$$\mathcal{A}_p := \mu \frac{NT\mathcal{T}e^{1-N}}{c_\lambda}\left(\frac{8Np}{\varepsilon\rho}\right)^N, \quad \mathcal{B}_p := \mu \frac{\mathcal{A}Te^{1-N}}{(\varepsilon\rho)^{(2+N)}c_\lambda} 2^{3N+6}N^{4+N}p^{2+N} \tag{38}$$

one finds, for all $s \ge 2$,

$$\left\| v_i^{[s]} \right\|_{(1-d_s)\varepsilon\rho} \le \mathcal{A}_p\chi^s + \mathcal{B}_p \sum_{r=1}^{s-1} \tilde{\alpha}_s \gamma_{s-r} =: \gamma_s.$$

Let us now substitute $\tilde{\alpha}_s$ as described in Prop. 4.1. This gives rise to the following majorising recurrence relations

$$\begin{aligned}
\gamma_2 &= \tilde{\mathcal{A}}_p\chi^2 \\
&\dots \\
\gamma_\sigma &= \tilde{\mathcal{A}}_p\chi^\sigma \\
\gamma_{\sigma+1} &= \tilde{\mathcal{A}}_p\chi^{\sigma+1} + \mathcal{B}_p\alpha^\sigma\gamma_1 \\
&\dots \\
\gamma_{2\sigma} &= \tilde{\mathcal{A}}_p\chi^{2\sigma} + \mathcal{B}_p\alpha^\sigma\gamma_\sigma \\
\gamma_{2\sigma+1} &= \tilde{\mathcal{A}}_p\chi^{2\sigma+1} + \mathcal{B}_p(\alpha^\sigma\gamma_{\sigma+1} + \alpha^{2\sigma}\gamma_1) \\
&\dots
\end{aligned} \tag{39}$$

where we have set

$$\tilde{\mathcal{A}}_p := \max\{\chi^{-1}\mathcal{K}_1, \mathcal{A}_p\} \tag{40}$$

in such a way such a sequence holds for the initial condition $\gamma_1 \equiv \tilde{\mathcal{A}}_p\chi$ as well, according to (32), and it is compatible with the case $\sigma = 1$.
In order to find the expression for the general term $\gamma_{s=j\sigma+k}$ of the sequence above, the so-called *generating function method* is used. See, e.g., [85].
Namely, after having defined the "generating function" parameterised by k

$$\mathscr{G}_k(w) := \sum_{j\ge0} \gamma_{j\sigma+k} w^{j\sigma}, \tag{41}$$

the method consists in multiplying each equation of (39) of the form $\gamma_{j\sigma+k}$ by $w^{j\sigma}$ then summing up both sides. This procedure leads to the following equation for $\mathscr{G}_k$

$$\mathscr{G}_k(w)\left(1 - \mathcal{B}_p\sum_{j\ge1}(\alpha w)^{j\sigma}\right) = \tilde{\mathcal{A}}_p\chi^k\sum_{j\ge0}(\chi w)^{j\sigma}.$$

In this case, it is sufficient to observe that there exists (a sufficiently small) $r_c > 0$ such that $\mathcal{B}_p\sum_{j\ge1}(\alpha w)^{j\sigma} \le 1/2$ for all $z \in B_{r_c}(0) \subset \mathbb{C}$, where $B_{r_c}(0)$ denotes the disk of radius r_c centred at the origin. This allows us to give an explicit expression for $\mathscr{G}_k(w)$, which reads as

$$\mathscr{G}_k(w) = \tilde{\mathcal{A}}_p\chi^k\left\{1 + \sum_{j\ge1}\left[\chi^{j\sigma} + \mathcal{B}_p\alpha^\sigma\sum_{l=1}^{j}\chi^{\sigma(j-l)}\left((1+\mathcal{B}_p)\alpha^\sigma\right)^{l-1}\right]\right\}.$$

This immediately yields, for all $j \ge 0$, the required expression (31). □

Proposition 5.1. *Define*

$$\mathscr{L} := 2^{9N+18}(N^2+2N)^{2(N+2)}e^{\rho\Gamma-2N-4}T\rho^{-2(N+3)}\Gamma^{-2-N}c_\lambda^{-1}. \tag{42}$$

Let us set $p := n_p\sigma \in \mathbb{N}$, suppose Γ as in assumption (5), and ε and μ to be chosen in a way

$$\mu^2 \mathscr{L} n_p^{(2+N)} e^{-\frac{\rho}{4\varepsilon}} \leq (2e^2)^{-1}, \qquad e^{-\frac{\rho}{2\varepsilon}} \leq (2e^2)^{-1}. \tag{43}$$

Then

$$\gamma_{j\sigma+k} \leq 4\tilde{\mathcal{A}}_p e^{-j-k}. \tag{44}$$

Proof. Firstly, it is immediate to check that assumption (5) on Γ implies that

$$\alpha, \chi \leq 1/e, \tag{45}$$

see (19) and (34), respectively.
As for the second term appearing in (31) the procedure is as follows. Recall (7), so we get from the second of definition (38) and (19),

$$\mathcal{B}_p\alpha^\sigma \leq \mu^2\left[2^{3N+7}N^{2N+4}T\rho^{-2N-3)}e^{\rho\Gamma}c_\lambda^{-1}\right]n_p^{2+N}\varepsilon^{-2(N+2)}e^{-\frac{\rho}{2\varepsilon}}. \tag{46}$$

By using an argument similar to the one used in the bound (21), one can easily prove that,

$$\varepsilon^{-2(N+2)}e^{-\frac{\rho}{2\varepsilon}} \leq \left(\frac{8(N+2)}{e\rho}\right)^{2(N+2)} e^{-\frac{\rho}{4\varepsilon}},$$

for all $\rho, \varepsilon > 0$, hence the r.h.s. of bound (46) approaches zero exponentially as $\varepsilon \to 0$. By substituting the previous bound into (46), using the definition (42) and the first of bounds (43), one gets

$$(1+\mathcal{B}_p)\alpha^\sigma \leq e^{-2}. \tag{47}$$

On the other hand, bound (45) holds, hence

$$\sum_{l=1}^{j}\chi^{\sigma(j-l)}\left((1+\mathcal{B}_p)\alpha^\sigma\right)^{l-1} \leq \sum_{l=1}^{j}e^{1-j+\sigma(j-l)} = e^{-(\sigma+2)(1+j)}\frac{(e^{j\sigma}-e^{2j})}{(e^\sigma-e^2)} =: h(j,\sigma).$$

It it now possible to observe that, for all $\sigma, j \geq 1$ the following bound hold

$$h(j,\sigma) \leq (e-1)^{-1}e^3e^{-j}. \tag{48}$$

For this purpose, let us distinguish the cases $h(j,1) \leq [e^2/(e-1)]e^{-j}$ then $h(j,2) \leq ee^{-j}$ and finally, for all $\sigma \geq 3$,

$$h(j,\sigma) \leq (e^\sigma - e^2)^{-1}(e^{\sigma+2})e^{-j} \leq (e-1)^{-1}e^3e^{-j},$$

It is now sufficient to observe that $(e-1)^{-1}e^3$ constitutes the largest bound as coefficient of e^{-j} so that bound (48) is proven. Hence, by using (45), (47) and (48) in (31), the bound (44) immediately follows. □

Lemma 5.2. *Let us set*

$$n_p = \left\lfloor \left(\frac{e^{\frac{\rho}{4\varepsilon}}}{2\mu^2 e^2 \mathscr{L}} \right)^{\frac{1}{2+N}} \right\rfloor , \tag{49}$$

noticing that $n_p \geq 1$ for sufficiently small ε. Then, there exists a $O(1)$ constant, $\mathscr{S}$, such that the remainder is bounded as follows

$$\max_{t\in[0,T]} \int_{\mathbb{T}^N} |\boldsymbol{v}(\boldsymbol{x},t)|^2_{\varepsilon\rho/2}\, d\boldsymbol{x} \leq \frac{4(2\pi)^N T^5}{r^-} e^{\frac{1}{r^-}} \mathscr{S}^2 \exp\left[-\left(\frac{e^{\frac{\rho}{4\varepsilon}}}{2\mu^2 e^2 \mathscr{L}} \right)^{\frac{1}{2+N}} \right] . \tag{50}$$

Note that the latter statement completes the proof of the main theorem, in particular it is sufficient to set $\tilde{\mathscr{S}} := 4(2\pi)^N T^5 \mathscr{S}^2 \exp(1/r^-)/r^-$ and $\tilde{\mathscr{E}} := (2\mu^2 e^2 \mathscr{L})^{-1}$ to get (8). Furthermore, as it is necessary to guarantee that $n_p \geq 1$, it will be sufficient to require that

$$\varepsilon \leq \varepsilon_0 := \frac{\rho}{4\log(2\mu^2 e^2 \mathcal{L})}, \tag{51}$$

which implies, *a fortiori*, the second of (43), provided that

$$\mu^2 \mathcal{L} \geq 1. \tag{52}$$

Remark 5.1. *A choice similar to (49) is the key step of the analytic part of the Nekhoroshev Theorem. It relies on the fact that, despite the perturbative series are divergent, there exists an optimal normalisation order which minimises the remainder in the Hamiltonian normal form. This step turns out to be the key ingredient to obtain the celebrated exponentially small size of such a remainder. See, e.g., [86] and [80].*

Remark 5.2. *A comment is in order regarding bound (52). Despite the case $\mu^2\mathcal{L} < 1$ is a possibility, it would require μ to be "extremely" small, given the size of $\mathcal{L}$, see definition (42). As a consequence, it would be hardly relevant in the applications. In fact, such a particular case, could even suggest a totally different perturbative approach. Hence, we are going to "restrict" ourselves to the most natural scenario given by bound (52), (and most likely $\mu^2\mathcal{L} \gg 1$).*

Proof. Let us recall the expression of $F^{[p]}$ given in (17). First of all, we have

$$\begin{aligned} \left\| \sum_{q=0}^{p-1} \sum_{r=0}^{p-q-1} R^{[p-r]} \partial_{tt} v_i^{[r+q]} \right\|_{(1-d_p)\varepsilon\rho} &\leq 2\mathcal{A} \sum_{q=0}^{p-1} \sum_{r=0}^{p-q-1} \{(\varepsilon\rho)[(1-d_{r+q})-(1-d_p)]\}^{-2} \tilde{\alpha}_{p-r}\gamma_{r+q} \\ &\leq 8\mathcal{A}p^2(\varepsilon\rho)^{-2} \sum_{q=0}^{p-1} \sum_{r=0}^{p-q-1} \tilde{\alpha}_{p-r}\gamma_{r+q} \end{aligned} \tag{53}$$

By using a similar procedure, one finds

$$\left\| \sum_{r=1}^{p-1} R^{[r]} \partial_{tt} v_i^{[p-r]} \right\|_{(1-d_p)\varepsilon\rho} \leq 8p^2 \mathcal{A}(\varepsilon\rho)^{-2} \sum_{r=1}^{p-1} \tilde{\alpha}_r \gamma_{p-r}. \tag{54}$$

As for the remaining terms appearing in equations (17), the following bounds hold

$$\left\| \partial_{x_j} C_{ijhk}^{[p-r]} \right\|_{(1-d_p)\varepsilon\rho} \leq 2/(\rho\varepsilon)\tilde{\alpha}_{p-r}$$
$$\left\| \partial_{x_k} v_h^{[r+q]} \right\|_{(1-d_p)\varepsilon\rho} \leq 2p/[\rho\varepsilon(p+q-r)]\gamma_{r+q}$$
$$\left\| \partial^2_{x_j x_k} v_h^{[r+q]} \right\|_{(1-d_p)\varepsilon\rho} \leq 4p^2/[\rho\varepsilon(p+q-r)]^2\gamma_{r+q}$$

implying that

$$\left\| \sum_{q=0}^{p-1} \sum_{r=0}^{p-q-1} \sum_{j=1}^{N} \left(\sum_{h,k=1}^{N} C_{ijhk}^{[p-r]} \partial_{x_k} v_h^{[r+s]} \right) \right\|_{(1-d_p)\varepsilon\rho} \leq \frac{16p^2 \mathcal{A} N^3}{\rho^2\varepsilon^2} \sum_{q=0}^{p-1} \sum_{r=0}^{p-q-1} \tilde{\alpha}_{p-r}\gamma_{r+q}. \tag{55}$$

Similarly,

$$\left\| \sum_{r=1}^{p-1} \sum_{j=1}^{N} \left(\sum_{h,k=1}^{N} C_{ijhk}^{[r]} \partial_{x_k} v_h^{[p-r]} \right) \right\|_{(1-d_p)\varepsilon\rho} \leq \frac{16p^2 \mathcal{A} N^3}{\rho^2\varepsilon^2} \sum_{r=1}^{p-1} \tilde{\alpha}_r \gamma_{p-r}. \tag{56}$$

Let us now set $p = n_p\sigma$ where $n_p \geq 1$ by hypothesis. Furthermore, hypotheses (45) and (44) imply

$$\sum_{r=1}^{n_p\sigma-1} \tilde{\alpha}_r \gamma_{n_p\sigma-r} = \sum_{j=1}^{n_p-1} \alpha^{j\sigma}\gamma_{\sigma(n_p-j)} \leq 4\tilde{\mathcal{A}}_p \sum_{j=1}^{n_p-1} e^{-[n_p+j(\sigma-1)]} \leq 8e\tilde{\mathcal{A}}_p e^{-\sigma+n_p},$$

furthermore

$$\begin{aligned}
\sum_{q=0}^{n_p\sigma-1} \sum_{r=0}^{n_p\sigma-q-1} \tilde{\alpha}_{n_p\sigma-r}\gamma_{r+q} &= \sum_{q=1}^{n_p} \tilde{\alpha}_{q\sigma} \sum_{l=(n_p-q)\sigma}^{n_p\sigma-1} \gamma_l \\
&= 4\tilde{\mathcal{A}}_p \sum_{q=1}^{n_p} e^{-(n_p+\sigma q-q)} \sum_{k=0}^{n_p-1} e^{-k} \\
&\leq 16e\tilde{\mathcal{A}}_p e^{-(\sigma+n_p)},
\end{aligned}$$

where the first equality is easily proven by induction and the inequality $\sum_{k=0}^{+\infty} e^{-k} < 2$ has been used in the last passage.
By using the above obtained bounds and bounds (45), one has

$$\left\| \boldsymbol{G}^{[p]} + \boldsymbol{F}^{[p]} \right\|_{(1-d_p)\varepsilon\rho} \leq eN \left[\mathcal{T} + 192(\varepsilon\rho)^{-2}\mathcal{A}\tilde{\mathcal{A}}_p(1+2N^3) \right] e^{-(n_p+\sigma)} \tag{57}$$

obtained via the elementary bound $\chi^{n_p\sigma} \leq e^{-n_p\sigma} \leq e^{1-n_p-\sigma}$.
After having recalled the expressions for $\mathcal{A}$ and $\tilde{\mathcal{A}}_p$ as in (19) and (40), respectively, one has that

$$192(\varepsilon\rho)^{-2}\mathcal{A}\tilde{\mathcal{A}}_p(1+2N^3) = \left[3\mu^2 2^{2N+7} N^{2N+1}(1+2N^3)(C^* e^{\Gamma\rho} + T\mathcal{T}c_\lambda^{-1})e^{\Gamma\rho}\rho^{-(N+2)} \right] n_p^N \varepsilon^{-2N}$$

Hence, by using the latter and the following bounds,

$$\varepsilon^{-2(N+1)}\varepsilon^{-\frac{1}{\varepsilon\Gamma}} \leq \left[2e^{-1}(N+1)\Gamma \right]^{2(N+1)}, \qquad n_p^N e^{-n_p} \leq \left[2e^{-1}N \right]^N e^{-\frac{n_p}{2}},$$

it is possible to define $\mathscr{S}$ as

$$\mathscr{S} := e^{\frac{3}{2}} N \left[\mathcal{T} + 3\mu 2^{2N+7} N^{2N+1} (1 + 2N^3) \left(C^* e^{\Gamma\rho} + T\mathcal{T} c_\lambda^{-1} \right) \frac{e^{\Gamma\rho}}{\rho^{N+2}} \left[\frac{2(N+1)\Gamma}{e} \right]^{3N+2} \right], \quad (58)$$

in order to get the estimate

$$\left\| \boldsymbol{G}^{[p]} + \boldsymbol{F}^{[p]} \right\|_{(1-d_p)\varepsilon\rho} \leq \mathscr{S} e^{-\frac{n_p}{2}} / \sqrt{e} \leq \mathscr{S} \exp \left[-\frac{1}{2} \left(\frac{e^{\frac{\rho}{4\varepsilon}}}{2\mu^2 e^2 \mathscr{L}} \right)^{\frac{1}{2+N}} \right].$$

where assumption (49) has been used in the last passage (recall that $(1 - d_p) = 1/2$ by construction). The proof is complete by invoking Prop. 4.2 where $\tilde{\beta}$ is set as the last term of the previous bound. □

Conclusions and future development

In this paper, a scheme borrowed from the classical perturbation theory is extended to an infinite dimensional model arising from elasto-dynamics. The algorithm, which exploits the spectral decay of real-analytic functions, consists of a hierarchy, the lowest order of which naturally represents the fully homogenised model.

The “optimal” choice of the normalisation order, typical of the Nekhoroshev Theorem approach, has been used here for the determination of n_p, and has represented the main ingredient in order to obtain "extremely good" approximation of the solutions, being the error superexponentially small in the cell dimension.

It is worthwhile to emphasize the full constructivity of the scheme at hand, which can be implemented simply by following the steps described in the flow-chart depicted in Fig. 2. For practical applications this can be carried out without addressing the quantitative part described in Secs. 4 and 5. More importantly, as it is common in this kind of results, the threshold of validity for ε, i.e. ε_0, is expected to be way smaller than the actual one. That is why analytic estimates as those performed in Sec. 5 are often performed in conjunction with computer assisted tools, in order to obtain more realistic values of ε_0.

Future developments will concern a detailed validation of the proposed spectro-hierarchical approach outside the mentioned threshold, ε_0, through numerical simulations and experiments, and the estimation of its accuracy with respect to traditional asymptotic or variational-asymptotic homogenization schemes.

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